Linear Algebra II

07/04/2014, Monday, 9:00-12:00

You are **NOT** allowed to use any type of calculators.

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1 (15 pts) **Gram-Schmidt process**

Consider the vector space P_2 with the inner product

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\,\mathrm{d}x.$$

Apply the Gram-Schmidt process to transform the basis $\{1, x, x^2\}$ into an orthonormal basis.

2 (7+8=15 pts) **Cayley-Hamilton theorem**

(a) Consider the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}.$$

Find real numbers a and b such that $(M - aI)(M - bI) = 0$.

(b) Consider the matrix

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Find real numbers a and b such that $(M - aI)(M - bI) = 0$.

3 (3+9+3=15 pts) **Singular value decomposition**

Consider the matrix

$$M = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}.$$

(a) Show that the singular values of M are $\sigma_1 = 5$ and $\sigma_2 = 3$.

(b) Find a singular value decomposition for M.

(c) Find the best rank 1 approximation of M.

4 (10+5=15 pts) **Eigenvalues and singular values**

Let A be an $n \times n$ matrix.

(a) Let λ be a real eigenvalue of A. Show that

$$\sigma_n \leqslant |\lambda| \leqslant \sigma_1$$

where σ_1 and σ_n are the largest and the smallest singular values of A, respectively.

(b) Suppose that A is symmetric. Show that $|\lambda|$ is a singular value of A if λ is an eigenvalue of A.

5 (10+5=15 pts) **Positive definiteness**

(a) Consider the function

$$f(x, y) = x^3 + y^3 - 3xy.$$

Find the stationary points of f and determine whether its stationary points are local minimum/maximum or saddle points.

(b) Let

$$M = \begin{bmatrix} a & -a & 0 \\ -a & b & a \\ 0 & a & a \end{bmatrix}$$

where a and b are real numbers. Determine all values of a and b for which M is positive definite.

6 (2+3+10=15 pts) **Jordan canonical form**

Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

(a) Find the eigenvalues of A.

(b) Is A diagonalizable? Why?

(c) Put A into the Jordan canonical form.

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10 pts free